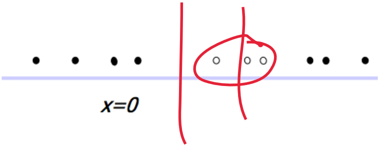
Machine Learning: Chenhao Tan

University of Colorado Boulder

LECTURE 11

Slides adapted from Jordan Boyd-Graber

Can you solve this with linear separator?



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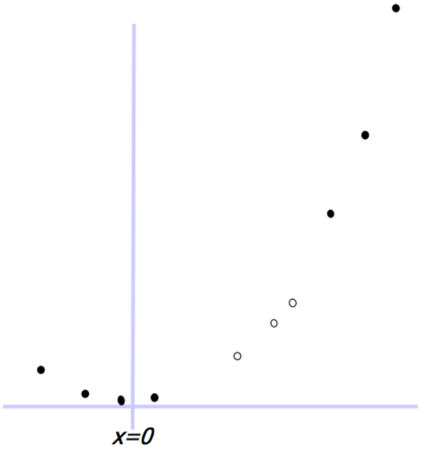

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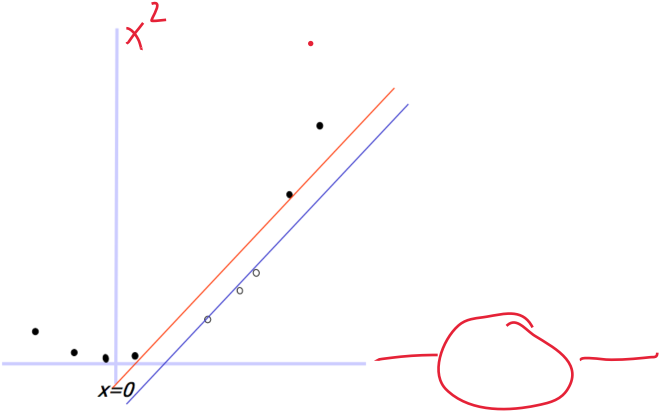
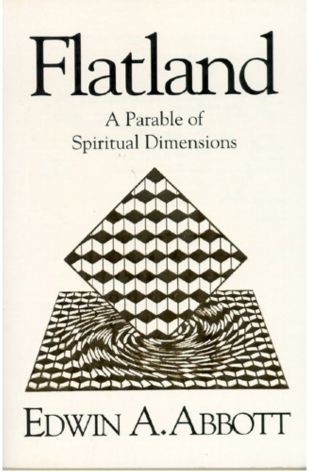


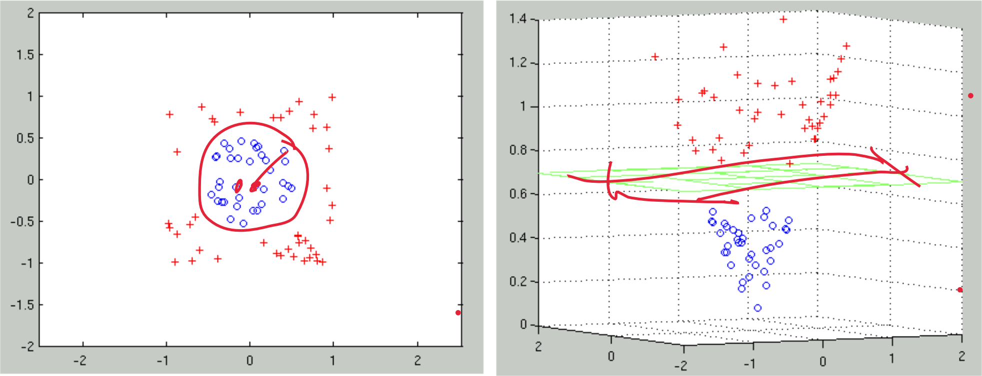
Adding another dimension



Behold yon miserable creature. That Point is a Being like ourselves, but confined to the non-dimensional Gulf. He is himself his own World, his own Universe; of any other than himself he can form no conception; he knows not Length, nor Breadth, nor Height, for he has had no experience of them; he has no cognizance even of the number Two; nor has he a thought of Plurality, for he is himself his One and All, being really Nothing. Yet mark his perfect self-contentment, and hence learn this lesson, that to be self-contented is to be vile and ignorant, and that to aspire is better than to be blindly and impotently happy.

Problems get easier in higher dimensions

$$(x_1, x_2) \Rightarrow (x_1, x_2, \sqrt{x_1^2 + x_2^2})$$



What's special about SVMs?

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{i=1}^{m} \alpha_i \alpha_j y_i y_j (\boldsymbol{x_i} \cdot \boldsymbol{x_j}) \tag{1}$$

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- Kernels!

What's a kernel?

- A function $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ is a kernel over $\mathcal{X}$.
- This is equivalent to taking the dot product $\langle \phi(x_1), \phi(x_2) \rangle$ for some mapping
- **Mercer's Theorem**: So long as the function is continuous and symmetric, then K admits an expansion of the form

$\phi(x_i) = (x_i, x_i^2)$

$$K(x, x') = \sum_{n=0}^{\infty} a_n \phi_n(x) \phi_n(x') \tag{2}$$

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- The computational cost is just in computing the kernel

Kernel Matrix

The important property of the kernel matrix $\boldsymbol{K} = [K(x_i, x_j)]_{ij} \in \mathbb{R}^{m \times m}$ is symmetric positive semidefinite.

$$K = \begin{pmatrix} K(x_1, x_1) & \cdots & K(x_1, x_m) \\ \vdots & & \vdots \\ K(x_m, x_1) & \cdots & K(x_m, x_m) \end{pmatrix}$$

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Also known as Gram matrix.

(x_i, x_i^2)

$c = 0 \quad d = 2$

Polynomial Kernel

$$K(x, x') = (x \cdot x' + c)^d \tag{3}$$

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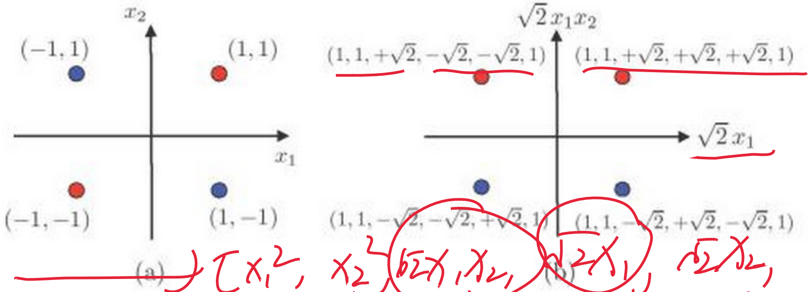
When $d = 2, c = 1$:



Gaussian Kernel

RBF

$$-\gamma \|x' - x\|^2$$

$$K(x, x') = \exp - \frac{\|x' - x\|^2}{2\sigma^2} \quad (4)$$

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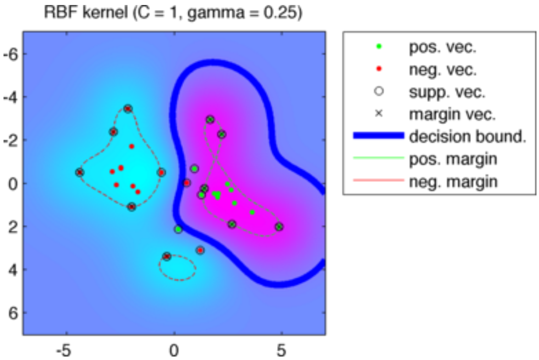

Tree Kernels

- Sometimes we have example x that are hard to express as vectors
- For example sentences “a dog” and “a cat”: internal syntax structure

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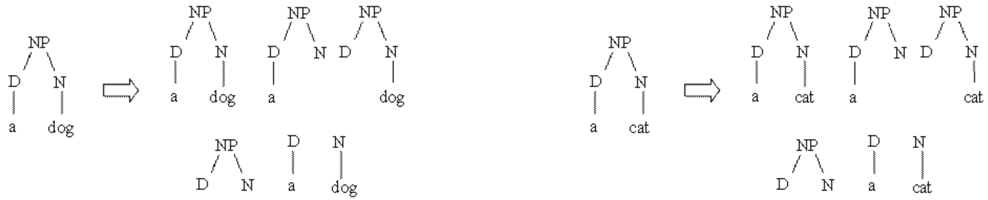

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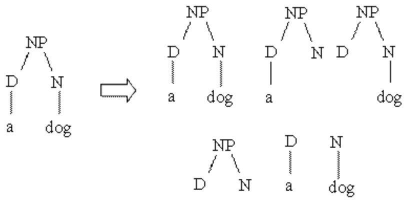
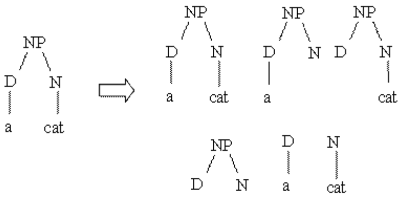




$3/5$ structures match, so tree kernel returns $.6$

What does this do to learnability?

- Kernelized hypothesis spaces are obviously more complicated
- What does this do to complexity?

How does it affect optimization

- Replace all dot product with kernel evaluations $K(x_1, x_2)$
- Makes computation more expensive, overall structure is the same
- Try linear first!

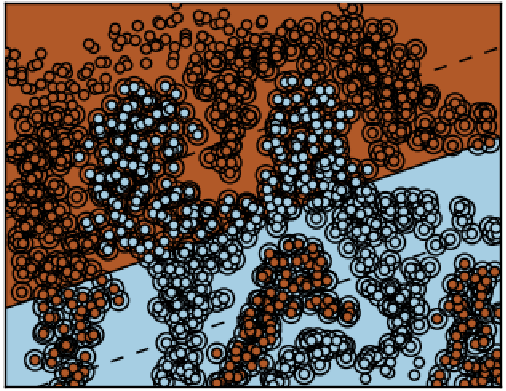
Outline

Examples

Kernelized SVM

```
X, Y = read_data("ex8a.txt")
clf = svm.SVC(kernel=kk, degree=dd, gamma=gg)
clf.fit(X, Y)
```

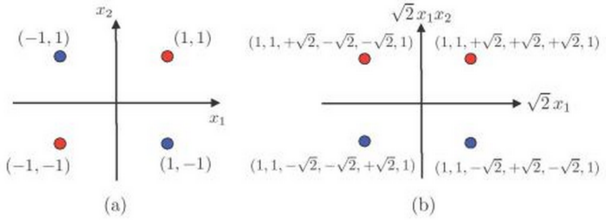
Linear Kernel Doesn't Work



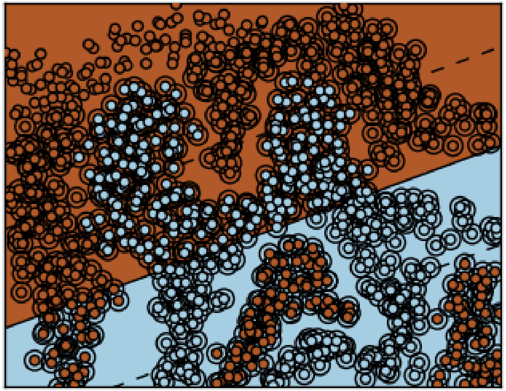
Polynomial Kernel

$$K(x, x') = (x \cdot x' + c)^d \tag{6}$$

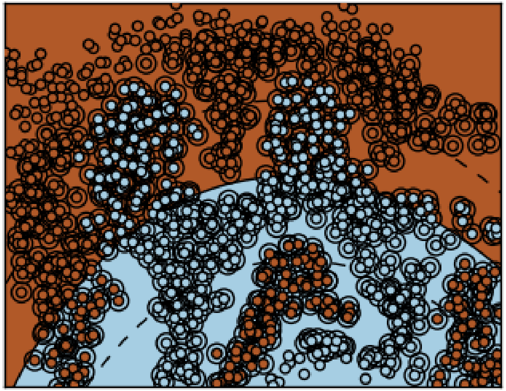
When $d = 2$:



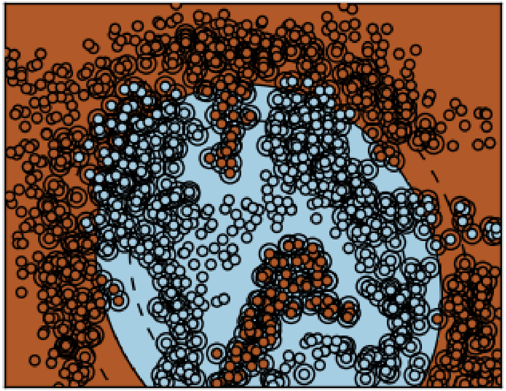
Polynomial Kernel $d = 1, c = 5$



Polynomial Kernel $d = 2, c = 5$

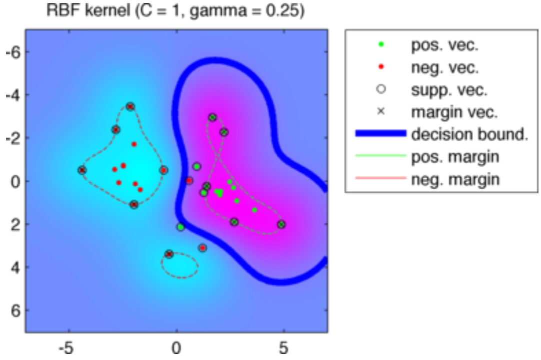


Polynomial Kernel $d = 3, c = 5$

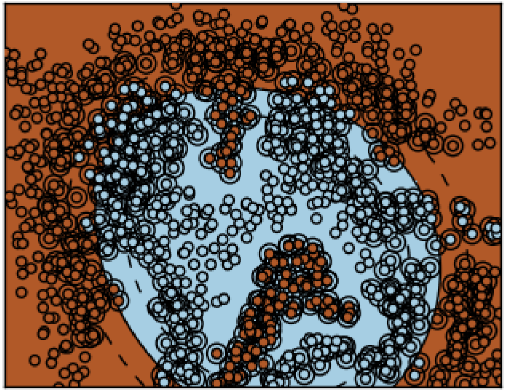


Gaussian Kernel

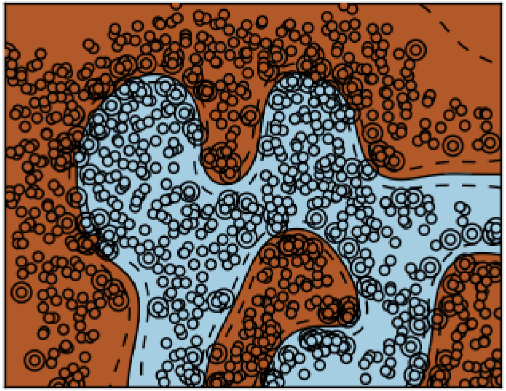
$$K(x, x') = \exp\left(\gamma \left\|x' - x\right\|^2\right) \tag{7}$$



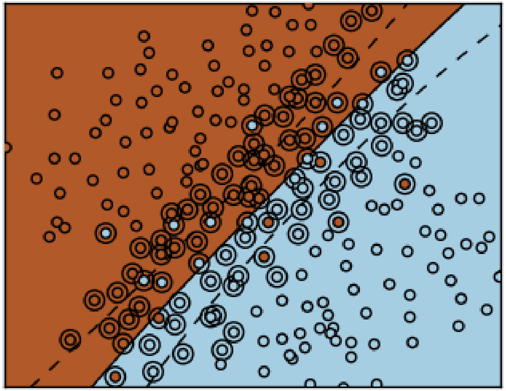
RBF Kernel $\gamma = 2$



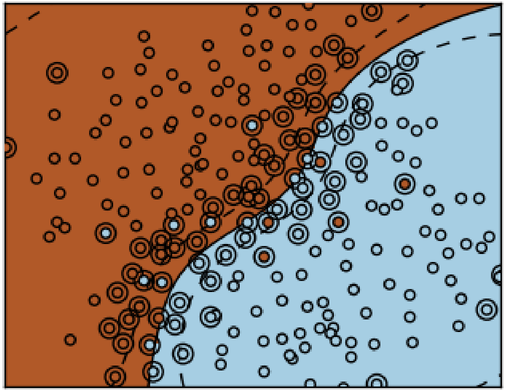
RBF Kernel $\gamma = 100$



RBF Kernel $\gamma = 1$



RBF Kernel $\gamma = 10$



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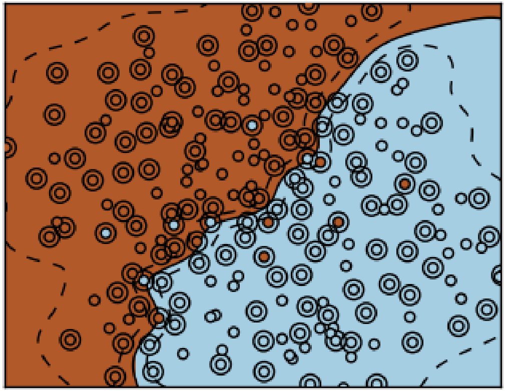
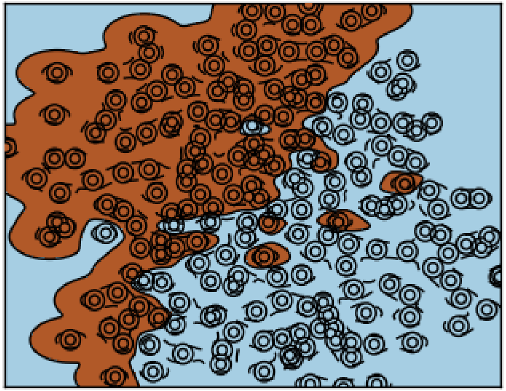


RBF Kernel $\gamma = 1000$



Be careful!

- Which has the lowest training error?
- Which one would generalize best?

Recap

- This completes our discussion of SVMs
- Workhorse method of machine learning
- Flexible, fast, effective

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- Workhorse method of machine learning
- Flexible, fast, effective
- Kernels: applicable to wide range of data, inner product trick keeps method simple